Generalized Maximum Principle for Higher Order Ordinary Differential Equations

QIN SHENG AND RAVI P. AGARWAL

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, 0511, Singapore

Submitted by Murray H. Protter

Received September 9, 1991

The purpose of this note is to establish the following generalized maximum principle.

THEOREM. *Let $1 \leqslant k \leqslant n-1$ $(n \geqslant 2)$ and $u \in C^{(n)}[a, b]$ be such that*

$$u^{(n)}(x) \geqslant 0, \qquad x \in (a, b), \tag{1}$$

$$(-1)^{n-k} u^{(i)}(a) \geqslant 0, \qquad i = 1, 2, ..., k-1 \text{ (if such } i \text{ exist)}, \tag{2}$$

$$(-1)^{n-k+j} u^{(j)}(b) \geqslant 0, \qquad j = 1, 2, ..., n-k-1 \text{ (if such } j \text{ exist)}. \tag{3}$$

Then, in the case $n-k$ even u attains its minimum and in the case $n-k$ odd u attains its maximum either at a or b.

If $n = 2$, then obviously $k = 1$. In this case our theorem gives the classical maximum principle "if $u \in C^{(2)}[a, b]$ and $u''(x) \geqslant 0$, $x \in (a, b)$ and attains its maximum at an interior point of $[a, b]$, then u is identically a constant on $[a, b]$." The proof of this maximum principle and several of its applications are available in Protter and Weinberger [5].

For $n = 4$, $k = 2$ our theorem reduces to the maximum principle of Chow, Dunninger, and Lasota [3]. It reads as "if $u \in C^{(4)}[a, b]$ and satisfies the inequalities

$$u^{(4)}(x) \geqslant 0, \qquad x \in (a, b),$$

$$u'(a) \geqslant 0, \qquad u'(b) \leqslant 0,$$

then u attains its minimum at a or b." An alternative proof of this maximum principle and its applications to fourth order boundary value problems has been given by Kuttler [4].

0022-247X/93 $5.00

For $n=2m$, $k=m$ our theorem becomes the maximum principle of Šeda [6], which is stated as "if $u \in C^{(2m)}[a, b]$ and satisfies the inequalities

$$\begin{aligned} u^{(2m)}(x) &\geqslant 0, && x \in (a, b), \\ (-1)^m u^{(i)}(a) &\geqslant 0, && i = 1, 2, ..., m-1 \text{ (if such } i \text{ exist)}, \\ (-1)^{m+j} u^{(j)}(b) &\geqslant 0, && j = 1, 2, ..., m-1) \text{ (if such } j \text{ exist)}, \end{aligned}$$

then in the case m even (m odd) u attains its minimum (maximum) at either a or b." In the same paper, Šeda also separately proves two more particular cases of our theorem namely when $k=n-1$ and $k=1$.

Thus, our theorem accommodates several known maximum principles. Further, the novelty is in its proof which is rather constructive and does not require any sign property of Green's functions of the related boundary value problems, cf. [1].

Proof of the Theorem. To prove our theorem we need the following:

LEMMA. *Let $1 \leqslant k \leqslant n-1$ $(n \geqslant 2)$ and $u \in C^{(n)}[a, b]$. Then,*

$$u(x) = H_{n-1}(x) + \frac{1}{n!}(x-a)^k (x-b)^{n-k} u^{(n)}(\xi), \qquad x \in [a, b], \tag{4}$$

where $\xi \in (a, b)$, and $H_{n-1}(x)$ is the two-point Hermite interpolating polynomial of degree $n-1$ satisfying

$$H_{n-1}^{(i)}(a) = u^{(i)}(a), \qquad i = 0, 1, ..., k-1, \tag{5}$$

$$H_{n-1}^{(j)}(b) = u^{(j)}(b), \qquad j = 0, 1, ..., n-k-1. \tag{6}$$

Further, this interpolating polynomial $H_{n-1}(x)$ can be written as

$$H_{n-1}(x) = \sum_{i=0}^{k-1} c_i(x) u^{(i)}(a) + \sum_{j=0}^{n-k-1} d_j(x) u^{(j)}(b), \tag{7}$$

where $c_i(x)$, $i = 0, 1, ..., k-1$, and $d_j(x)$, $j = 0, 1, ..., n-k-1$, are polynomials of degree $n-1$, and appear as

$$c_i(x) = \frac{(b-a)^i}{i!} \left(\frac{x-b}{a-b}\right)^{n-k} \left(\frac{x-a}{b-a}\right)^i P_{k-i-1}(x), \qquad i = 0, 1, ..., k-1, \tag{8}$$

$$d_j(x) = \frac{(a-b)^j}{j!} \left(\frac{x-a}{b-a}\right)^k \left(\frac{x-b}{a-b}\right)^j Q_{n-k-j-1}(x), \qquad j = 0, 1, ..., n-k-1, \tag{9}$$

and

$$P_{k-i-1}(x) = \sum_{r=0}^{k-i-1} \binom{n-k+r-1}{r} \left(\frac{x-a}{b-a}\right)^r,$$

$$Q_{n-k-j-1}(x) = \sum_{\tau=0}^{n-k-j-1} \binom{k+\tau-1}{\tau} \left(\frac{x-b}{a-b}\right)^\tau.$$

The proof of this lemma is well known, e.g., see Berzin and Zhidkov [2, pp. 145–148].

It follows that, if $n-k$ is even, then in view of (1) representation (4) gives

$$u(x) \geqslant H_{n-1}(x), \qquad x \in [a, b]. \tag{10}$$

Next, from (8) and (9) it is clear that for all $x \in [a, b]$, $c_i(x) \geqslant 0$, $0 \leqslant i \leqslant k-1$, and $(-1)^j d_j(x) \geqslant 0$, $0 \leqslant j \leqslant n-k-1$. Therefore, conditions (2) and (3) in (7) lead to

$$H_{n-1}(x) \geqslant c_0(x)\, u(a) + d_0(x)\, u(b), \qquad x \in [a, b]. \tag{11}$$

Combining (10) and (11), we find that

$$u(x) \geqslant c_0(x)\, u(a) + d_0(x)\, u(b), \qquad x \in [a, b]. \tag{12}$$

Let $u(x) \equiv 1$ in (4), then in view of (7) we find that

$$1 = c_0(x) + d_0(x), \qquad x \in [a, b]. \tag{13}$$

Using (13) in (12), we obtain

$$\begin{aligned} u(x) &\geqslant (c_0(x) + d_0(x)) \min\{u(a), u(b)\} \\ &= \min\{u(a), u(b)\}, \qquad x \in [a, b]. \end{aligned} \tag{14}$$

The proof for the case $n-k$ is odd is identical.

References

1. R. P. Agarwal, "Boundary Value Problems for Higher Order Differential Equations," World Scientific, Singapore, 1986.
2. I. S. Berzin and N. P. Zhidkov, "Computing Methods," Pergamon, Oxford, 1956.
3. S. N. Chow, D. R. Dunninger, and A. Lasota, A maximum principle for fourth order ordinary differential equations, *J. Differential Equations* **14** (1973), 101–105.

4. J. R. KUTTLER, A remark on the paper, "A maximum principle for fourth order ordinary differential equations" by Chow, Dunninger and Lasota, *J. Differential Equations* **17** (1975), 44–45.
5. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1967.
6. V. ŠEDA, Two remarks on boundary value problems for ordinary differential equations, *J. Differential Equations* **26**, No. 2 (1976), 278–290.